Design and Analysis of Algorithms

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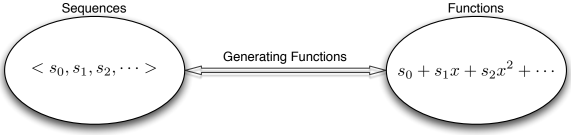
Solving the recurrence equations

There are different approaches to do this:

- Constructing Recursion Tree
- Performing Substitution
- Using Induction
- Master Theorem
- Generating Functions

Generating Functions

Generating Functions transform problems about sequences into problems about functions where we have powerful Mathematical tools.



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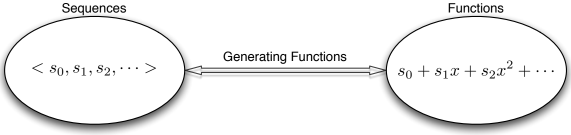

$$< s_0, s_1, s_2, s_3, \cdots > \quad \longleftrightarrow \quad F(x) = \sum_{i=0}^{\infty} s_i x^i$$

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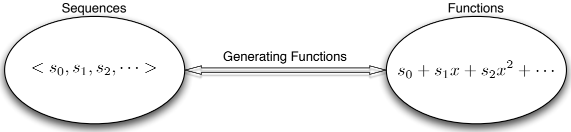

$$< s_0, s_1, s_2, s_3, \cdots > \quad \longleftrightarrow \quad F(x) = \sum_{i=0}^{\infty} s_i x^i$$

Examples:

$$< 1, 1, 1, 1, \cdots > \quad \longleftrightarrow \quad 1 + z + z^2 + z^3 + \cdots = \frac{1}{1 - z}$$

Generating Functions

Example

- $< 0, 0, 0, 0, \cdots > \quad \longleftrightarrow \quad 0 + 0x + 0x^2 + 0x^3 + \cdots = 0.$
- $< 1, 0, 0, 0, \cdots > \quad \longleftrightarrow \quad 1 + 0x + 0x^2 + 0x^3 + \cdots = 1.$
- $< 3, 2, 1, 0, \cdots > \quad \longleftrightarrow \quad 3 + 2x + x^2 + 0x^3 + \cdots = 3 + 2x + x^2.$
- $< 1, -1, 1, -1, \cdots > \quad \longleftrightarrow \quad 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}.$
- $< 1, a, a^2, a^3, \cdots > \quad \longleftrightarrow \quad 1 + ax + a^2x^2 + a^3x^3 + \cdots = \frac{1}{1-ax}.$
- $< 1, 0, 1, 0, \cdots > \quad \longleftrightarrow \quad 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}.$

Operations on Generating Functions

Scaling

if

$$< f_0,\, f_1,\, f_2,\, f_3,\, \cdots > \quad \longleftrightarrow \quad F(x),$$

then

$$< cf_0,\, cf_1,\, cf_2,\, cf_3,\, \cdots > \quad \longleftrightarrow \quad cF(x).$$

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Addition

if

$$< f_0,\, f_1,\, f_2,\, f_3,\, \cdots > \quad \longleftrightarrow \quad F(x)$$

and

$$< g_0,\, g_1,\, g_2,\, g_3,\, \cdots > \quad \longleftrightarrow \quad G(x),$$

then

$$< f_0 + g_0,\, f_1 + g_1,\, f_2 + g_2,\, f_3 + g_3,\, \cdots > \quad \longleftrightarrow \quad F(x) + G(x).$$

Operations on Generating Functions

Example

$$< 1,\ 0,\ 1,\ 0,\ \cdots > \quad \longleftrightarrow \quad 1 + x^2 + x^4 + \cdots = \frac{1}{1 - x^2}$$

$$\Longrightarrow \frac{2}{1 - x^2} = 2 + 2x^2 + 2x^4 + \cdots$$

Operations on Generating Functions

Example

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Example

$$< 1,\ 1,\ 1,\ 1,\ \cdots > \quad \longleftrightarrow \quad \frac{1}{1-x}$$

and

$$< 1,\ -1,\ 1,\ -1,\ \cdots > \quad \longleftrightarrow \quad \frac{1}{1+x}$$

$$\Longrightarrow < 2,\ 0,\ 2,\ 0,\ \cdots > \quad \longleftrightarrow \quad \frac{1}{1-x} + \frac{1}{1+x} = \frac{2}{1-x^2}$$

Operations on Generating Functions

Right Shifting

if

$$< f_0,\, f_1,\, f_2,\, f_3,\, \cdots > \quad \longleftrightarrow \quad F(x),$$

then

$$< \underbrace{0,\, 0,\, \cdots,\, 0}_{k-\text{zeros}},\, f_0,\, f_1,\, f_2,\, f_3,\, \cdots > \quad \longleftrightarrow \quad x^k F(x).$$

Operations on Generating Functions

Right Shifting

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Derivative

if

$$< f_0,\ f_1,\ f_2,\ f_3,\ \cdots > \quad \longleftrightarrow \quad F(x)$$

then

$$< f_1,\ 2f_2,\ 3f_3,\ 4f_4,\ \cdots > \quad \longleftrightarrow \quad \frac{d}{dx}F(x).$$

Operations on Generating Functions

Example

- $< 1, 1, 1, 1, \cdots > \quad \longleftrightarrow \quad \frac{1}{1-x}$
- $< 1, 2, 3, 4, \cdots > \quad \longleftrightarrow \quad \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$
- $< 0, 1, 2, 3, \cdots > \quad \longleftrightarrow \quad x.\frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$
- $< 1, 4, 9, 16, \cdots > \quad \longleftrightarrow \quad \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3}$
- $< 0, 1, 4, 9, \cdots > \quad \longleftrightarrow \quad x.\frac{1+x}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}$

Operations on Generating Functions

Product

if

$$< a_0,\ a_1,\ a_2,\ a_3,\ \cdots > \quad \longleftrightarrow \quad A(x)$$

and

$$< b_0,\ b_1,\ b_2,\ b_3,\ \cdots > \quad \longleftrightarrow \quad B(x),$$

then

$$< c_0,\ c_1,\ c_2,\ c_3,\ \cdots > \quad \longleftrightarrow \quad A(x).B(x),$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0.$$

Note that $C(x) = A(x).B(x) = \sum_{n=0}^{\infty} c_n x^n$

The Fibonacci Sequence

Definition

The Fibonacci sequence is defined by the following recurrence equation:

$$f_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2. \end{cases}$$

The generating function for this sequence can be considered as follows:

$$F(x) \;=\; f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots \;=\; \sum_{n=0}^{\infty} f_n x^n$$

The Fibonacci Sequence

First approach: By expanding the recurrence equation, the following sequence is obtained:

$$< 0,\ 1,\ f_1 + f_0,\ f_2 + f_1,\ f_3 + f_2,\ \cdots >$$

One can break this sequence into a sum of three sequences as follows:

$$\begin{array}{rlllll}
< 0, & 1, & 0, & 0, & 0, & \cdots > \longleftrightarrow x \\
+ < 0, & f_0, & f_1, & f_2, & f_3, & \cdots > \longleftrightarrow xF(x) \\
+ < 0, & 0, & f_0, & f_1, & f_2, & \cdots > \longleftrightarrow x^2F(x) \\
\hline
< 0, & 1 + f_0, & f_1 + f_0, & f_2 + f_1, & f_3 + f_2, & \cdots > \longleftrightarrow x + xF(x) + x^2F(x)
\end{array}$$

$$\Longrightarrow F(x) = x + xF(x) + x^2F(x) \Longrightarrow F(x) = \frac{x}{1 - x - x^2}$$

The Fibonacci Sequence

Second approach:

$$\begin{aligned}
f_n &= f_{n-1} + f_{n-2} \\
&\Downarrow \\
x^n f_n &= x^n f_{n-1} + x^n f_{n-2} \\
&\Downarrow \\
\sum_{n=2}^{\infty} x^n f_n &= \sum_{n=2}^{\infty} x^n f_{n-1} + \sum_{n=2}^{\infty} x^n f_{n-2} \\
&\Downarrow \\
\sum_{n=0}^{\infty} x^n f_n - x &= x \left[\sum_{n=0}^{\infty} x^n f_n \right] + x^2 \left[\sum_{n=0}^{\infty} x^n f_n \right] \\
&\Downarrow \\
F(x) - x &= xF(x) + x^2 F(x) \\
&\Downarrow \\
F(x) &= \frac{x}{1 - x - x^2}
\end{aligned}$$

The Fibonacci Sequence

Now, we should expand $F(x)$ to extract coefficients...

$$1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x) \Longrightarrow \alpha_1 = \frac{1}{2}(1 + \sqrt{5}),\ \alpha_2 = \frac{1}{2}(1 - \sqrt{5})$$

Next we do as follows:

$$\frac{x}{1 - x - x^2} = \frac{A_1}{1 - \alpha_1 x} + \frac{A_2}{1 - \alpha_2 x} \Longrightarrow A_1 = \frac{1}{\sqrt{5}},\ A_2 = -\frac{1}{\sqrt{5}}$$

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Now we have:

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \alpha_1 x} - \frac{1}{1 - \alpha_2 x} \right) \\ &= \frac{1}{\sqrt{5}} \left((1 + \alpha_1 x + {\alpha_1}^2 x^2 + \cdots) - (1 + \alpha_2 x + {\alpha_2}^2 x^2 + \cdots) \right) \end{aligned}$$

so

$$f_n = \frac{{\alpha_1}^n - {\alpha_2}^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right).$$

Exercise

1. Suppose that $S_n = \{1, 2, 3, \cdots, n\}$. An involution over the set S_n is a permutation $\pi : S_n \mapsto S_n$ of order at most 2 (i.e. $1 \leq \forall i \leq n$, $\pi^2(i) = i$). Derive a recurrence equation to count the number of involution for a set of size n and then try to solve it using Generating Functions.
2. Try to obtain a closed form for the following recurrence equation:

$$p_n = \begin{cases} 1 & \text{if } n \leq 2, \\ 2 & \text{if } n = 3, \\ p_{n-1} + (n-1)p_{n-2} - p_{n-3} + p_{n-4} & \text{if } n \geq 4. \end{cases}$$

Fast Multipication

Suppose that x and y are integers. The size of x is assumed to be n, where n is the number of bits required to represent x (n=$\lceil Log_2(x) \rceil$). Computing $z = x.y$:

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  - Best case: $O(n)$ time complexity.
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- Direct approach: has $O(n^2)$ time complexity.
- Consider the 1-valued bits:
  - Best case: $O(n)$ time complexity.
  - Worse case: $O(n^2)$ time complexity.
- Divide and Conquer:

Since $x = a.2^{n/2} + b$ and $y = c.2^{n/2} + d$, so

$$z = x.y = a.c.2^n + (a.d + b.c).2^{n/2} + b.d$$

In this case we have $T(n) = 4T(n/2) + O(n) = O(n^2)$! How we can reduce the time complexity?

Fast Multipication

In order to improve the algorithm and reduce the time complexity, we use the following trick:

- $w_1 = a + b$
- $w_2 = c + d$
- $u = w_1.w_2 = a.c + a.d + b.c + b.d$
- $v = a.c$
- $w = b.d$

Now, we have:

$$z = x.y = v.2^n + (u - v - w).2^{n/2} + w$$

and so

$$T(n) = 3T(n/2) + O(n) = \Theta(n^{Log_2(3)}).$$

Fast Multipication

```
Fast_Multiplication(x, y, n){
    if(n > 1){
        a ← MSB(x);
        b ← LSB(x);
        c ← MSB(y);
        d ← LSB(y);
        w1 ← ADD(a, b, n/2);
        w2 ← ADD(c, d, n/2);
        u ← Fast_Multiplication(w1, w2, n/2);
        v ← Fast_Multiplication(a, c, n/2);
        w ← Fast_Multiplication(b, d, n/2);
        res ← Shift(v, n);
        res ← Add(res, Shift(SUB(SUB(u, v, n/2), w, n/2), n/2), 2n);
        res ← Add(res, w, 2n);
        return(res);
    }
}
```

Exercises

1. Try to solve the Fast Multiplication by dividing each number into three parts and analyze it.