به نام او

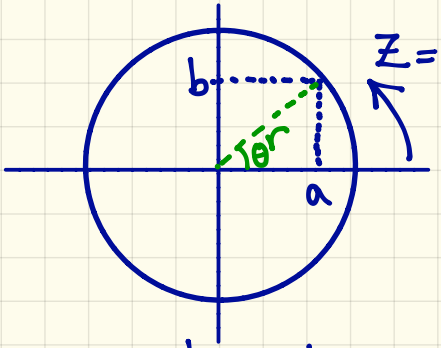
ریاضی عمومی ۱

اعداد مختلط - بخش سوم

$Z = a + ib \qquad a, b \in \mathbb{R}$

$a = \operatorname{Re} Z$ قسمت حقیقی Z

$b = \operatorname{Im} Z$ قسمت موهومی Z



$Z = a + ib$

$|Z| = r = \sqrt{a^2 + b^2}$

$b = 0 \Rightarrow |Z|$: قدرمطلق در $\mathbb{R}$

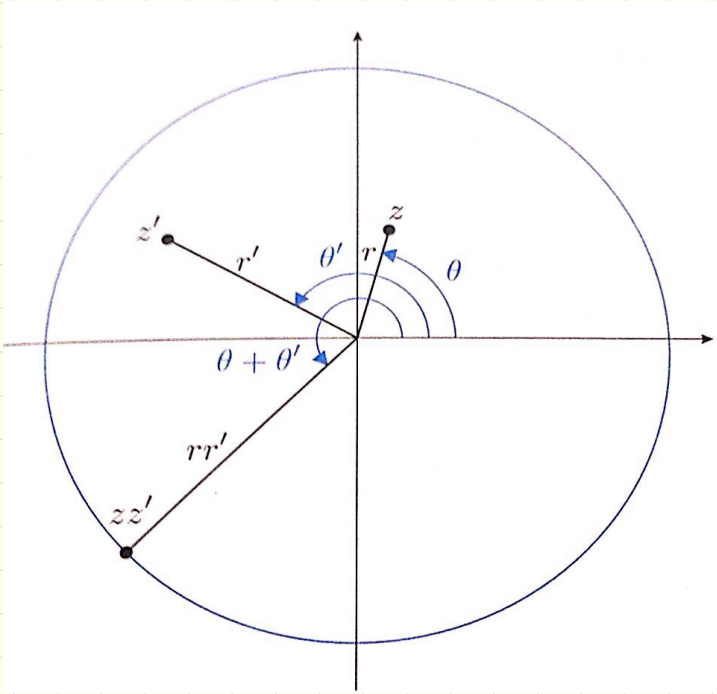
$\theta : \arg Z$

$Z = a + ib = \underbrace{r(\cos\theta + i\sin\theta)}_{r \operatorname{cis}\theta} : (r, \theta) = (|Z|, \arg Z)$

$\operatorname{cis}\theta = \cos\theta + i\sin\theta$

$Z_1 = r_1 \operatorname{cis}\theta_1, \quad Z_2 = r_2 \operatorname{cis}\theta_2 : Z_1 Z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

$z_1 = r_1 \operatorname{cis}\theta_1, \quad z_2 = r_2 \operatorname{cis}\theta_2 : z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

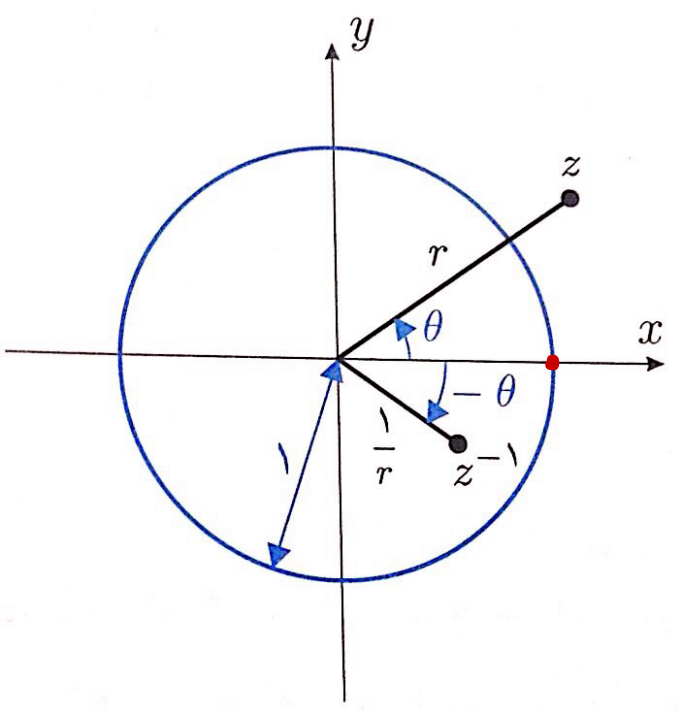


$$|zz'| = |z||z'|$$

$$\arg(zz') = \arg z + \arg z'$$

مثال: z ، $\frac{1}{z}$

$1 = 1 \times \operatorname{cis}(0)$



مثال: $z^n \overset{?}{=} 1$

$$z = r\,\mathrm{cis}\,\theta$$

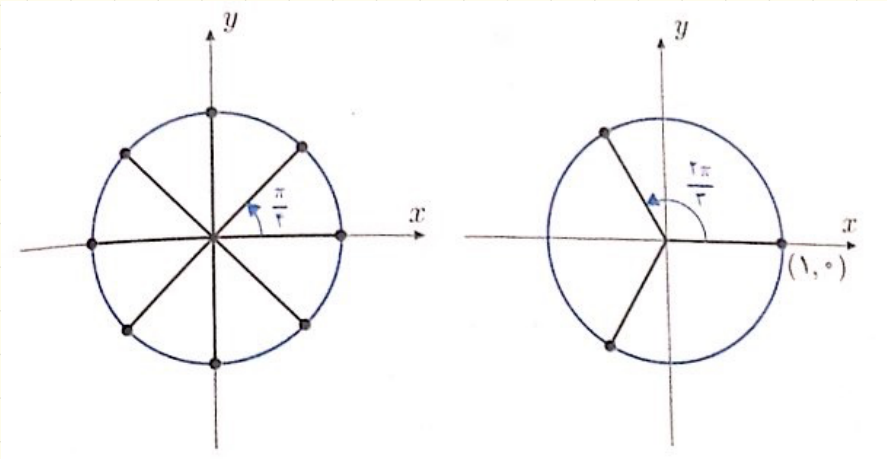
$$1\,\mathrm{cis}\,0 = 1 = z^n = r^n\,\mathrm{cis}\,n\theta$$

$r \in \mathbb{R}^+$ $\quad \underbrace{r^n = 1}_{\Downarrow} \quad n\theta = 0 + 2k\pi \quad k \in \mathbb{Z}$

$$r = 1 \qquad \theta = \frac{2k\pi}{n} \qquad 1 \times \mathrm{cis}\,\frac{2k\pi}{n} \quad k \in \mathbb{Z}$$

$k = 0, \dots, n-1$

$$z^n - 1 = (z - \alpha_1) \dots (z - \alpha_n)$$



$$Z_0 = Z^n \qquad Z_0 = r_0 \operatorname{cis}\theta_0$$

$$\sqrt[n]{r_0} \times \operatorname{cis}\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right) \qquad k = 0, \dots, n-1$$

$$= \sqrt[n]{r_0} \operatorname{cis}\frac{\theta_0}{n} \times \underbrace{\operatorname{cis}\frac{2k\pi}{n}}_{\text{ریشه } n \text{ واحد}}$$

$$Z = r\operatorname{cis}\theta \Rightarrow Z^n = r^n \operatorname{cis} n\theta \qquad n \in \mathbb{Z}$$

$$\frac{1}{Z} = \frac{1}{r}\operatorname{cis}(-\theta)$$

n عدد صحیح منفی : $Z^n = (Z^{-1})^{-n}$

$$u_1 = \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4}{27}p^3}\right)} \qquad v_1 = \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4}{27}p^3}\right)}$$

$$u_1 \times v_1 = -\frac{p}{3} \qquad \omega = \operatorname{cis}\frac{2\pi}{3}$$

$$u_2 = \omega_1 u_1 \qquad v_2 = \omega_2 u_2 \qquad (\omega_1)^3 = (\omega_2)^3 = 1$$

$$u_2 \times v_2 = -\frac{p}{3} \Rightarrow \omega_1 \omega_2 = 1 \qquad \omega_1, \omega_2 = 1$$

$$\omega_1 = \omega \qquad \omega_2 = \omega^2 \qquad \omega_1 = \omega^2, \omega_2 = \omega$$

مثال: $t^3 - 6t + 4 = 0$, $p = -6$, $q = 4$

$$\frac{1}{2}\left(-4 \pm \sqrt{16-32}\right) = -2 \pm 2i$$

$$u^3 = -2+2i = 2\sqrt{2}\left(\operatorname{cis}\left(\frac{3\pi}{4}\right)\right) \Rightarrow u = \sqrt{2}\operatorname{cis}\frac{\pi}{4} \times \omega_1$$

$$v^3 = -2-2i = 2\sqrt{2}\left(\operatorname{cis}\frac{5\pi}{4}\right) \Rightarrow v = \sqrt{2}\operatorname{cis}\frac{5\pi}{12} \times \omega_2$$

یک ریشه سوم واحد

$$-\frac{p}{3} = 2 = uv = 2 \times \overbrace{\operatorname{cis}\underbrace{\left(\frac{\pi}{4}+\frac{5\pi}{12}\right)}_{\frac{8\pi}{12}=\frac{2\pi}{3}}}^{\omega} \times \omega_1\omega_2$$

$$\omega\,\omega_1\omega_2 = 1 \quad : \quad \omega_1 = \omega \Rightarrow \omega_2 = \omega \quad , \quad \omega_1 = 1 \quad \omega_2 = \omega^2$$

$$\omega_1 = \omega^2 \Rightarrow \omega_2 = 1$$

$$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$u = \sqrt{2}\operatorname{cis}\frac{\pi}{4}\operatorname{cis}\frac{2\pi}{3} \qquad v = \sqrt{2}\operatorname{cis}\frac{5\pi}{12}\operatorname{cis}\frac{2\pi}{3}$$

$$u = \sqrt{2}\ \text{cis}\,\frac{\pi}{4}\ \text{cis}\,\frac{2\pi}{3} \qquad v = \sqrt{2}\,\text{cis}\,\frac{5\pi}{12}\ \text{cis}\,\frac{2\pi}{3}$$

$$u+v = \sqrt{2}\ \text{cis}\,\frac{2\pi}{3}\left(\text{cis}\,\frac{\pi}{4} + \text{cis}\,\frac{5\pi}{12}\right) = \sqrt{2}\left(\text{cis}\left(\frac{8\pi+3\pi}{12}\right) + \text{cis}\left(\frac{8\pi+5\pi}{12}\right)\right)$$

$$= \sqrt{2}\left[\text{cis}\left(\frac{11\pi}{12}\right) + \text{cis}\left(\frac{13\pi}{12}\right)\right] = 2\sqrt{2}\ \text{cos}\,\frac{11\pi}{12} = -2\sqrt{2}\ \text{cos}\,\frac{\pi}{12} = -1-\sqrt{3}$$